ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Computational Geometry 41 (2008) 206–218
Computational Geometry
Theory and Applications
www.elsevier.com/locate/comgeo

Improving the topology computation of an arrangement of cubics

Jorge Caravantes, Laureano Gonzalez-Vega [*,1]

Departamento de Matematicas, Estadistica y Computacion Universidad de Cantabria, Spain

Received 8 August 2007; received in revised form 24 February 2008; accepted 5 March 2008

Available online 19 March 2008

Communicated by M. Sharir

Abstract

This paper is devoted to improve the efficiency of the algorithm introduced in [A. Eigenwillig, L. Kettner, E. Schömer, N. Wolpert, Exact, efficient and complete arrangement computation for cubic curves, Computational Geometry 35 (2006) 36–73] for analyzing the topology of an arrangement of real algebraic plane curves by using deeper the well-known geometry of reducible cubics instead of relying on general algebraic tools.

Keywords: Curve arrangements; Topology; Cubic curves; Algebraic methods

Introduction

The computation of the topology of an arrangement of real algebraic plane curves is a natural generalization of the well-known problem in Computational Geometry dealing with the determination of the cell decomposition for an arrangements of lines and segments in the plane. Arrangements of real algebraic plane curves appear very often when dealing with practical questions in Computer Aided Geometric Design such as the surface-to-surface intersection problem.

The computational study of the topology for a single real algebraic plane curve has been widely studied by adapting to this concrete case the so called Cylindrical Algebraic Decomposition algorithm: see for example [10] or [5] and the references contained there in. Efficient algorithms for arrangements of straight segments can be found in [12], [7] and [13] and for conics in [14] and [1]. In [6] the authors introduced a complete, exact and efficient algorithm for computing the topology of an arrangement of cubic curves by adapting the Bentley–Ottmann sweep-line method to this situation through the use of limited algebraic machinery (mainly resultants, subresultants and real root determination for univariate polynomials). This algorithm proceeds by analyzing first every single cubic, then by studying every pair of cubics in the arrangement and, finally, by merging all the available information.

The main goal of this paper is a complete modification of the analysis of a single cubic in the algorithm introduced in [6] when the considered cubic has at least two singularities. In this case, the considered cubic is known to be

* Corresponding author.
E-mail addresses: jorge.caravantes@unican.es (J. Caravantes), laureano.gonzalez@unican.es (L. Gonzalez-Vega).
1 Partially supported by the Spanish MEC grant MTM2005-08690-C02-02.

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2008.03.001

reducible and this fact is used to simplify many of the calculations proposed in [6]. The proposed improvement comes from using deeper the well-known geometry of the reducible cubics instead of relying on general algebraic tools. This idea has also been used in [3] to extend the algorithm in [6] for analyzing arrangements of cubic curves to study in a similar way the topology of an arrangement of quartic curves.

The structure of this paper is as follows. After presenting in the first section some algebraic and geometric preliminaries devoted to polynomials, subresultants and the geometry of cubic curves, the second section briefly reviews the algorithm in [6] paying special attention to the part of the algorithm we are going to improve. The third section contains the detailed description of our proposal for the analysis of a single reducible cubic and the last section presents several examples together with a report of the performed experimentation showing how the proposed modification improves the overall efficiency of the algorithm introduced in [6].

1. Algebraic and geometric preliminaries

Throughout this section we introduce one of the main tools to be used in what follows, subresultants, together with their main property and the definition of the squarefree decomposition of an univariate polynomial. Later we recall the definitions of the relevant points of the curve to be analyzed whose study will guide the algorithm providing the final topological answer.

Definition 1.1. Let

$$P=\sum_{i=0}^{m} a_i y^{m-i} \quad \text{and} \quad Q=\sum_{i=0}^{n} b_i y^{n-i}$$

be two polynomials in y with coefficients in a field $\mathbb{K}$ (in our case $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). We define the jth subresultant polynomial of P and Q with respect to the variable y in the following way (as in [11]):

$$\mathbf{Sres}_j(P,Q;y)=(-1)^{j(m-j+1)}\left|\begin{matrix} a_0 & a_1 & a_2 & \dots & \dots & a_m & & & \\ & \ddots & \ddots & \ddots & & & \ddots & & \\ & & a_0 & a_1 & a_2 & \dots & \dots & a_m & \\ & & & & & 1 & -y & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 1 & -y \\ b_0 & b_1 & b_2 & \dots & \dots & \dots & b_n & & \\ & \ddots & \ddots & \ddots & & & & \ddots & \\ & & b_0 & b_1 & b_2 & \dots & \dots & \dots & b_n \end{matrix}\right| \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n-j \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} j \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m-j \end{matrix}$$

and we define the kth subresultant coefficient of P and Q with respect to y, $\mathbf{sres}_k(P,Q;y)$, as the coefficient of y^k in $\mathbf{Sres}_k(P,Q;y)$. Finally, the resultant of P and Q with respect to y is:

$$\mathbf{res}(P,Q;y)=\mathbf{Sres}_0(P,Q;y)=\mathbf{sres}_0(P,Q;y).$$

There are several different ways of computing efficiently subresultants, different from their definition: see, for example, [2] and the references contained therein. Next we introduce the main property of subresultants that will be used in what follows. A proof can be found, for example, in [2].

Theorem 1.1. *Let f and g be two polynomials in $\mathbb{K}[y]$. Then the following two facts are equivalent*:

- $\mathbf{sres}_i(f,g;y)=0$ *for all* $i<k$ *and* $\mathbf{sres}_k(f,g;y)\neq 0$.
- *The greatest common divisor of f and g has degree k and is equal to* $\mathbf{Sres}_k(f,g;y)$ (*up to multiplication by nonzero elements of* $\mathbb{K}$).

Next we introduce some notations and properties of polynomials to be used in what follows:

- Given $f \in \mathbb{R}[x, y]$, the leading coefficient of f with respect to the variable y will be denoted by $l(f)$, which is a polynomial in $\mathbb{R}[x]$.
- A polynomial $R \in \mathbb{R}[x]$ is said to be squarefree if it is not divided by the square of any other polynomial or, equivalently, if it has no multiple roots (in $\mathbb{C}$).
- Given $R \in \mathbb{R}[x]$, there exist $R_1, \ldots, R_l \in \mathbb{R}[x]$ such that each R_k is squarefree and R_i and R_j are coprime for $i \neq j$ and

$$R = \prod_{i=1}^{l} R_i^i.$$

  This is the so called squarefree decomposition of R and there exist algorithms computing it by just performing some greatest common divisor computations (see [8], for example).

The geometric part of this section starts by introducing the definition of the points of a real algebraic plane curve that deserve analysis and a known property of cubic curves. Let f and g be two polynomials in two variables x and y with real coefficients and we write f_x and f_y to denote the partial derivatives of f. Then:

- We denote

$$V(f) = \{(\alpha, \beta) \in \mathbb{R}^2\colon\ f(\alpha, \beta) = 0\}, \qquad V_{\mathbb{C}}(f) = \{(\alpha, \beta) \in \mathbb{C}^2\colon\ f(\alpha, \beta) = 0\}.$$

- An *arc* of the curve $V(f)$ is the closure of a maximal connected subset of $V(f)$ which is the graph of an analytic function on x.
- A left (resp. right) *x-extremal point* of $V(f)$ is a point P in $V(f)$ for which all arcs of $V(f)$ through P are to the right (resp. left) of P (e.g. $(0, 0)$ in $V(y^2 - x)$ is a left x-extremal point).
- A *singularity* is a point P in $V(f)$ such that $f_x(P) = f_y(P) = 0$.
- A *node* is an order two singularity through which exactly two arcs of the curve pass through. If these two arcs are real and the tangent lines to them at P are different then it is a *crunode*, if the tangent lines coincide then it is a *tacnode*, and if the two arcs are imaginary then it is an *acnode*. A *cusp* is an order two singularity where two arcs end.
- A singularity of a curve $V(f)$ defined by a polynomial $f \in \mathbb{Q}[x, y]$ is *locatable* if it has rational coordinates and these coordinates can be easily determined in terms of the coefficients of f. In the other case, we will call the singularity *unlocatable*.

Finally we present a well-known fact about the geometry of cubic plane curves with more than one singularity that is the main tool for the modification that we propose here to the algorithm in [6]. It can be found in basic textbooks on algebraic curves or Algebraic Geometry like, for example, [9].

Theorem 1.2. *Let f be a degree three squarefree polynomial in $\mathbb{R}[x, y]$ and $C = V(f)$ the corresponding cubic curve. Then*:

- *C has at most three singularities.*
- *If C has exactly two* (*complex*) *singularities then it consists of a line and a smooth conic* (*that can be two parallel lines*).
- *If C has three* (*complex*) *singularities then it consists of three lines.*

When C consists of a line and a conic or three lines, C is called reducible. *In other case it is irreducible* (*or non-reducible*).

Remark 1.1. Reducibility can happen in $\mathbb{C}[x, y]$ but not in $\mathbb{Q}[x, y]$. With the exception of Remark 4.1, reducibility is always considered in $\mathbb{C}$.

2. The algorithm overview

The method proposed in [6] to analyze the topology of an arrangement of plane cubic curves begins with the analysis of each curve separately. Since we propose a modification just for this step, we will briefly sketch this part of the whole method. Any property stated here is fully justified in [6].

The algorithm in [6] computes the topological analysis of a plane cubic $V(f)$, $f \in \mathbb{Q}[x, y]$, by assuming:

1. y-regularity, i.e. the coefficient of y^3 in $f(x, y)$ is a nonzero constant in $\mathbb{Q}$,
2. squarefreeness, i.e. there does not exist a polynomial h such that h^2 divides f,
3. no two points of $V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(f_y)$ are covertical, i.e. sharing the same x-coordinate,
4. no vertical flexes on $V(f)$, i.e. there is no point in $V(f)$ with vertical tangent $\{x = x_0\}$ such that $f(x_0, y)$ has a triple root, and
5. no vertical singularities on $V(f)$, i.e. no arc has vertical tangent at a singular point P of $V(f)$,

or detecting infractions to these conditions if they exist. These infractions (but the second one, which is solved by dividing by the corresponding greatest common divisor of f and f_y) are all avoidable after performing a linear change of coordinates (in a suitable way). So after an infraction is detected, we can restart again the algorithm after performing the corresponding linear change of coordinates.

We will now explain the algorithm, beginning with the general computations and finishing with the study of the events, i.e. the x-extremal points and the singularities.

2.1. General set up

The first condition is easy to check and the third one comes for free due to the degree of f. We now compute the resultant

$$R_f(x) = \mathbf{res}(f, f_y; y).$$

If $R_f(x) \equiv 0$ then the second condition is not satisfied, so we restart with the squarefree part of f, i.e.

$$\frac{f}{\gcd(f, f_y)}.$$

Next, we sort the real roots x_i of $R_f(x)$ and choose rational numbers r_i such that:

$$r_1 < x_1 < r_2 < x_2 < \cdots < r_n < x_n < r_{n-1}.$$

We also find the multiplicities m_i of all x_i through performing the squarefree decomposition

$$R_f(x) = \prod_{m=1}^{M} \left(R_{f,m}(x)\right)^m.$$

We know that, by the implicit function theorem and the y-regularity of f, everywhere but over the x_i, $V(f)$ has the shape of one or three disjoint graphs of functions, with this number being constant over each interval (x_{i-1}, x_i). Therefore, we order the $k_i \in \{1, 3\}$ (by definition) real roots of $f(r_i, y)$.

We finish by checking that there are no vertical flexes

$$v \in V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(f_y) \cap V_{\mathbb{C}}(f_{yy}) \setminus V_{\mathbb{C}}(f_x).$$

We have to check that the solutions x_i of the quadratic equation $\mathrm{res}(f_y, f_{yy}; y) = 0$ together with the multiple root y_i of $f_y(x_i, y)$ satisfy $f_x(x_i, y_i) = 0$. If the equation is identically zero (i.e. if $f_y = \frac{1}{2} f_{yy}^2$), checking that the real roots of $R_{f,2}(x)$ are real roots of $\mathrm{res}(f_x, f_{yy}; y)$ too.

We call a point $(x_i, y_i) \in V(f) \cap V(f_y)$ an *event point* and an arc of $V(f)$ containing it, *involved* in the event. We now have two possibilities for each event:

- $m_i = 1$, so the event point is a x-extremal point.
- $m_i > 1$, so (since vertical flexes are already discarded) the event point is a singularity.

2.2. x-extremal points

In this case we have $|k_i - k_{i+1}| = 2$ and we know if our point is a left x-extremal or right x-extremal point by just looking at which of k_i or k_{i+1} is greater. To check if the uninvolved arc passes above or below the event, we choose $j \in \{i, i+1\}$ such that $k_j = \min\{k_i, k_{i+1}\}$ and compute the second derivative

$$f_{yy}(r'_l, y) = ay - b$$

(with $a, b \in \mathbb{Q}$, $a \neq 0$), where r'_j is between r_j and x_i and is closer to x_i than any event x-coordinate of $V(f)$ or $V(f_y)$.

In this case it is well known that the point $(r'_j, \frac{b}{a})$ is on the same side of the event point as the uninvolved arc. Therefore, the uninvolved arc is above the event if

$$\operatorname{sign}\left(f\left(r'_j, \frac{b}{a}\right)\right) \neq \operatorname{sign}\bigl(l(f)\bigr)$$

and below otherwise. Further details and clarifying pictures can be found in [6]. Let this way of working be called in what follows *the second derivative trick*.

2.3. Singularities that can be explicitly located

When our event is a singularity and the corresponding squarefree part $R_{f,m_i}(x)$ has degree 1, both coordinates of the event point x_i and y_i are rational numbers and can be easily located (x_i is the only root of the linear polynomial $R_{f,m_i}(x)$, and y_i can be found by computing the greatest common divisor of $f(x_i, y)$ and $f_y(x_i, y)$ through subresultants). After this, we can compute y'_i, the only single root of $f(x_i, y)$. Comparing y_i and y'_i is enough to know the topology.

If we are interested also in the knowledge of the singularity and its type (as in [6]), we consider the polynomial $f(x - x_i, y - y_i)$ which, together with

$$f(x_i, y) = a(y - y_i)^2(y - y'_i),$$

gives all the needed information.

2.4. Other singularities

This is the step of the algorithm that we propose to modify in order to improve the overall efficiency of the method presented in [6].

When $\deg R_{f,m_i} > 1$, we cannot guarantee that our singularity is a point with rational coordinates, so we work without knowing explicitly the coordinates of the event point (x_i, y_i). First of all, and due to low degree, in this case it is well known that $m_i = 2$. We know that $(y - y_i)^2$ divides $f(x_i, y)$; let y'_i be the other root (which must be real). To know if the uninvolved arc is above or below the singularity (i.e. the sign of $y_i - y'_i$), we define a polynomial $\delta(x)$ whose value at x_i is precisely $y'_i - y_i$.

In [6] a method is proposed to compute $\delta(x)$ that we will not repeat here. This polynomial $\delta(x)$ is no longer used in this paper but it can be very useful when trying to generalize the algorithm in [6] to higher degrees. This is the reason why next we introduce a more efficient way than in [6] to compute it. Instead we introduce a different polynomial with a much more geometric flavour. If

$$\mathbf{Sres}_1(f, f_y; y) = a_1(x)y + a_2(x) = u(x, y)$$

then

$$u(x_i, y) = a_1(x_i)y + a_2(x_i) = \gcd\bigl(f(x_i, y), f_y(x_i, y)\bigr) = a_1(x_i)(y - y_i)$$

and therefore,

$$y_i = -\frac{a_2(x_i)}{a_1(x_i)}.$$

Then, the quotient $H(x, y)$ of the euclidean division of $f(x, y)$ and

$$l(f)\left(y+\frac{a_2(x)}{a_1(x)}\right)^2$$

with respect to y, which can be computed by applying twice Horner's Rule and ignoring the remainders, satisfies the property

$$H(x_i, y) = y - y_i'$$

for every x_i, root of $R_{f,2}(x)$. So the rational function

$$\delta(x) = H\left(x, -\frac{a_2(x)}{a_1(x)}\right)$$

verifies that

$$\delta(x_i) = y_i - y_i'$$

as we wanted, so the sign of $\delta(x_i)$ determines if the uninvolved arc is above or below the event as said before. Since $y_i - y_i'$ is finite and different from 0, this implies that x_i is not a real root of both the numerator and denominator of $\delta(x)$.

We finish by finding the sign of $\delta(x)$ in the real roots of $R_{f,2}(x)$ by computing these roots (even if they involve easy to manage square roots) and evaluating $\delta(x)$ at each real root if $\deg R_{f,2} = 2$. If $\deg R_{f,2} = 3$, we check the sign of $\delta(x_i)$ by evaluating $\delta(x)$ in a close enough rational number r' determined in the following way: we apply Descartes' Rule to the numerator $N(x)$ and denominator $D(x)$ of $\delta(x)$ to refine the isolating interval of x_i (as real root of $R_{f,2}$) until it is assured that there is no real roots of $N(x)$ and $D(x)$ in the refined isolating interval of x_i. Choosing any rational number in such interval provides the desired r'.

Remark 2.1. In the case when $\deg R_{f,2} = 3$ (i.e. three lines intersecting in three points), in order to check the relative position of the uninvolved arcs with respect to the event it is enough to proceed, as before described, with any of the three roots of $R_{f,2}$. With this information it is very easy to deduce what is happening over the two other x-coordinates (or real roots of $R_{f,2}$) (see [6] for more details).

3. Treating geometrically unlocatable singularities

Our idea starts from a fact that was mentioned in the last section: if $\deg R_{f,m_i} > 1$ then $m_i = 2$ and $\deg R_{f,2} \in \{2, 3\}$ and the shape of the cubic is determined by this degree. In the first case, $\deg R_{f,2} = 2$, our cubic is the union of a smooth conic (maybe two parallel lines) and a line. If $\deg R_{f,2} = 3$ then we are dealing with three lines.

Remark 3.1. The case $\deg R_{f,3} = 2$ is excluded because it corresponds to a conic curve intersecting a line in its two x-extremal points (so two vertical singularities arise in this case).

This section analyzes completely all the possible cases that will be treated separately. Namely:

- $\deg R_{f,2} = 3$ and it has one real root: we have a line and an acnode.
- $\deg R_{f,2} = 3$ and it has three real roots: we have three real lines.
- $\deg R_{f,2} = 2$ and it has no real roots: there are no event points to analyze.
- $\deg R_{f,2} = 2$ and $V(f)$ has two x-extremal points: we have a conic (ellipse or hyperbola) and a line.
- $\deg R_{f,2} = 2$ and $V(f)$ has one x-extremal point: a parabola and a line.
- $\deg R_{f,2} = 2$ and $V(f)$ has no x-extremal points: a hyperbola and a line.

Remark 3.2. The case $\deg R_{f,2} = 2$ and without x-extremal points cannot be a parabola and a line due to the y-regularity assumption.

Remark 3.3. When $\deg R_{f,2} = 2$, first of all, one has to check that $k_i = k_{i+1} = 3$ because otherwise there will be a vertical singularity and we will stop in order to perform a change of coordinates and restart. In the other cases, it is well known that both singularities are crunodes.

3.1. A real and two complex lines

If x_i is the only real root of $R_{f,m_i}(x)$ then it is the x-coordinate of an acnode and we can use the "*second derivative trick*" with any of both sides because (x_i, y_i) behaves for this matter as both left and right x-extremal point.

3.2. Three real lines

In this case we have that the only event points are the three singularities with x-coordinates x_1, x_2 and x_3. It is easy to check that the middle point in $V(f) \cap V(x - r_1)$ and the middle point in $V(f) \cap V(x - r_3)$ are in the same line $L \subset V(f)$ (let the other lines be L' and L''). Consider the second partial derivative f_{yy}. Then the line $V(f_{yy})$ cuts L (if they are parallel, see below 3.2.1) to the left or right of all the three singularities (i.e. when L is the middle arc) and the other two lines in the segments between singularities (see Fig. 2 in Example 4.2). So we compare if the only real root of $\mathbf{res}(f, f_{yy}; y)$ that is not in (x_1, x_3) is to the left or to the right side of the interval. If there is not such a root then $V(f_{yy})$ is parallel to L.

Let us suppose that it is to the left (since the other case is symmetric). Now we go to the infinity line (just taking the homogeneous component of degree three f_3 of f). We dehomogeneize by giving x the value 1, which means that the roots of $f_3(1, y)$ are the slopes of the three lines (and due to y-regularity, x does not divide $f_3(x, y)$, i.e. none of the three lines is vertical).

Remark 3.4. The fact that the roots α, β and γ of $f_3(1, y)$ are the slopes of the lines arises from the fact that they represent the points in infinity $(0:1:\alpha)$, $(0:1:\beta)$ and $(0:1:\gamma)$ (i.e. directions $(1, \alpha)$, $(1, \beta)$ and $(1, \gamma)$). A different way of seeing this is the following one: multiplying the equations of three lines:

$$y - \alpha x + a, \quad y - \beta x + b, \quad y - \gamma x + c$$

and checking that the roots of the degree three homogeneous component (when $x = 1$) are exactly α, β and γ.

We now compare the three real roots of f_3 (i.e. the slopes of L', L and L'', from lower to higher) with the slope of the line $V(f_{yy})$. There are two possibilities:

- The slope of L is bigger than the slope of the line $V(f_{yy})$. Since L cuts $V(f_{yy})$ to the left of the singularities, this means that L is above $V(f_{yy})$ during the events. Since L is the uninvolved arc in the middle singularity, we get that the uninvolved arc is above for the singularity whose x-coordinate is x_2 and below for the other two singularities.
- The slope of L is smaller than the slope of the line $V(f_{yy})$. Symmetric to the other case already analyzed.

3.2.1. $V(f_{yy})$ is parallel to L

If L and $V(f_{yy})$ are parallel then we have that there exists an unique $c \in \mathbb{R}$ so that $f_{yy} + c$ divides f. Next lemma shows how to compute explicitly such a c.

Lemma 3.1. *If*

$$f(x, y) = \sum_{i+j=0}^{3} a_{i,j} x^i y^j$$

and $a_{0,3} = 1$ *then the value of* c *such that* $f_{yy} + c$ *divides* f *is given by the formula*:

$$c = -2\frac{9a_{2,0} - 3a_{1,1}a_{1,2} - 3a_{0,2}a_{2,1} + 2a_{0,2}a_{1,2}^2}{a_{1,2}^2 - 3a_{2,1}}. \tag{1}$$

Proof. Since $f_{yy} = 6y + 2a_{1,2}x + 2a_{0,2}$ (remember that $a_{0,3} = 1$) intersects f in the infinity line, we get that the slope $-\frac{a_{1,2}}{3}$ is a root of $f_3(1, y)$. In order to compute c we now consider the remainder $r(x, c)$ of the division of f by $f_{yy} + c$ considered as polynomials in y. The coefficient of x^3 in $r(x, c)$ is $f_3(1, -\frac{a_{1,2}}{3})$, which vanishes in this case. The coefficient of x^2 in $r(x, c)$ is:

$$a_{2,0} + \frac{1}{18}a_{1,2}^2 c + \frac{2}{9}a_{0,2}a_{1,2}^2 - \frac{1}{3}a_{1,2}a_{1,1} - \frac{1}{3}a_{0,2}a_{2,1} - \frac{1}{6}ca_{2,1}.$$

This expression must vanish if $f_{yy}+c$ is a divisor of f. Thus the expression for c in the statement follows by solving this equation, provided that $a_{1,2}^2 - 3a_{2,1} \neq 0$. But the vanishing of such denominator, together with the vanishing of $f_3(1, a_{1,2})$ implies that $f_3(1, y) = (y - \frac{1}{3}a_{1,2})^3$ (i.e. f would consist of 3 parallel lines, which contradicts the case of three singularities which we are considering here). □

It is clear that $L \neq f_{yy}$, so c is obviously nonzero. Therefore, if $c > 0$ then L is below f_{yy}. Hence the middle singularity is above L (which is the uninvolved arc) and the two extremal singularities are below the uninvolved arc. The case $c < 0$ is symmetric.

3.3. The cubic has two x-extremal points

In this case, we will use the x-extremal points we studied before. There is again here a case distinction: consider the interval (a, b) whose end points are the x-coordinates of the x-extremal points. The possibilities are:

- Both singularities have x-coordinates in (a, b) (line and ellipse).
- Both singularities have x-coordinates to the left (or both to the right) of (a, b) (line and hyperbola, both singularities in the same branch of the conic).
- The interval defined by the x-coordinates of the singularities contains (a, b) (line and hyperbola, one singularity in each branch).

We explain the case of the ellipse since the other two cases are similar. We have an ellipse Q and a line L. Between the singularities, L is the arc in the middle and near the end points of the interval it is the uninvolved arc (so the upper or the lower one). Therefore, if the left x-extremal point has the uninvolved arc (i.e. L) above, in the left singularity, the arcs crossing are the upper ones (so the uninvolved arc is below the singularity). The other cases (L is below and to consider the right singularity) are symmetric (vertically or horizontally).

3.4. The cubic has one x-extremal point

In this case, the cubic consists of a parabola Q and a line L. This case is the same than the last one. We know the other extremal point, which is $x = \infty$ (let us use formally $+\infty$). So we solve this (locatable) extremal point and decide where is the uninvolved arc. Now, if the x-extremal point in the affine plane is a left one then we work as in the ellipse case. If it is a right x-extremal point then we work as in the case of two intersections in the left branch of a hyperbola.

3.5. The cubic has no x-extremal points

In this situation, the conic is a hyperbola (two parallel lines as a reducible case) which has clearly an upper branch and a lower one. It is impossible that $V(f_{yy})$ contains any of the singularities because a cubic polynomial with a double (and not triple) root cannot have a common root with its second derivative.

To determine the relative position of the uninvolved arc with respect to the event with x_i as x-coordinate, we take an x-coordinate r close enough to x_i (i.e. closer than any other x-coordinate of an event involving $V(f)$, $V(f_{yy})$ or both). It is not important if it is to the left or to the right of the singularity, since there are three arcs at both sides. Now we compute the unique real root q of $f_{yy}(r, y)$ (which is a linear polynomial) and compare the sign of the leading coefficient of f (with respect to the variable y) with the sign of $f(r, q)$. If they are equal then there are exactly two roots of $f(r, y)$ over q, so the singularity is over $V(f_{yy})$ and hence over the uninvolved arc. If the signs are opposite then we proceed symmetrically.

Note that this approach can be applied to any reducible cubic with crunodes, but in the other cases the approach previously explained seems more efficient, since they depend only on conditionals or easier calculations.

4. Examples and experimentation

Throughout this section we show the results of applying the method introduced in the previous section to several examples. First of all, we show how the algorithm is applied to a concrete example. Then we present several examples with several kind of unlocatable singularities and we close this section by comparing the efficiency of the proposed method with the one introduced in [6].

Example 4.1. The polynomial

$$f(x, y) = -3yx - 2x + 3y^2 - 2y - y^3 + 2y^2x - x^2y + 2x^2$$

is obviously y-regular and its squarefreeness is easy to check. First, we compute

$$f_y(x, y) = -3x + 6y - 2 - 3y^2 + 4yx - x^2$$

and

$$R_f(x) = 8x^5 - 79x^4 + 222x^3 - 131x^2 + 12x + 4.$$

Since $R_f(x)$ is not identically zero, we compute its squarefree decomposition

$$R_f(x) = (1 + 8x)(2 - 5x + x^2)^2 = R_{f,1}(x)\big(R_{f,2}(x)\big)^2$$

and we choose the rational numbers r_i separating the real roots of $R_f(x)$: $r_1 = -8$, $r_2 = 0$, $r_3 = 2$ and $r_4 = 7$. Therefore, fixed i, between r_i and r_{i+1} there is exactly one real root x_i of $R_f(x)$, and its multiplicity m_i is 1 for $i = 1$, and 2 for $i = 2$ and $i = 3$. The numbers k_i of arcs over each r_i are $k_1 = 1$ and $k_2 = k_3 = k_4 = 3$. Now it is easy to check that

$$V(f) \cap V(f_y) \cap V(f_{yy}) = \emptyset.$$

Now we begin to analyze the events starting with those of multiplicity 1 (i.e. the first one). So we take x_1. As explained before, we compare $k_1 = 1$ with $k_2 = 3$ and get that ours is a left x-extremal point. Since $V(f_y)$ is a hyperbola with no events, we can use r_1 in order to apply the “second derivative trick”. So we compute

$$f(r_1, y) = -42y + 144 - 13y^2 - y^3,$$

whose second derivative is $-26 - 6y$. This means that the flex point (of the graph $z = f(r_1, y)$) lays in $y = -13/3$. Since

$$f\left(r_1, \frac{13}{3}\right) = \frac{4408}{27} > 0$$

and the leading coefficient of $f(r_1, y)$ is negative, we have that the uninvolved arc over x_1 is above the event. Therefore we have a left x-extremal point and an arc below at the line $V(x - x_2)$.

To check what happens above or below the singularities, we notice that, since there is just one x-extremal point, the conic is a parabola. So first of all we look at the x-extremal point at the infinity (which is a right one, since the other one is left). In the line of infinity $V(f)$ is given by the homogeneous component of degree 3 of f:

$$f_3(x, y) = -y^3 + 2y^2x - x^2y = -y(x - y)^2.$$

So the slope of the line is zero (the single real root) and we have over the infinity x-coordinate a right x-extremal point and an arc below. Now we proceed like in the ellipse case and get that the uninvolved arc is below the event at the left singularity and above the event at the right one.

Fig. 1 shows, both, the curve $V(f)$ and their topology as produced by the `Maple` implementation of the algorithm presented in the previous section.

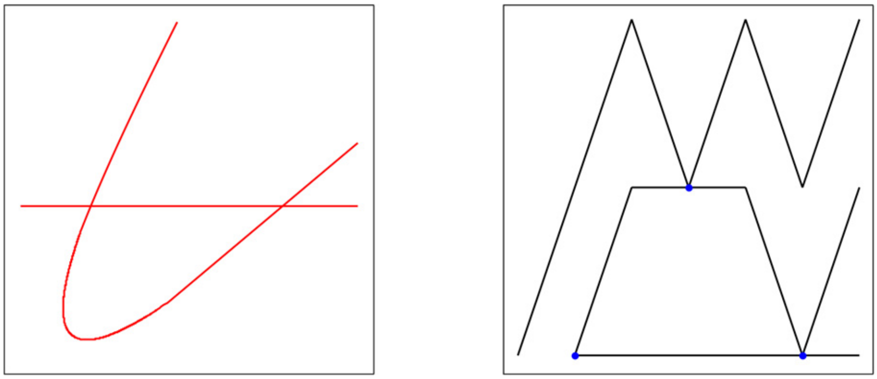

Fig. 1. The cubic curve $V(f)$ (left) and its topology (right).

The three following examples illustrate how the algorithm deals with different kinds of reducible cubics (i.e. cubics consisting on a line plus a conic or three lines) involving unlocatable singularities. The first one treats the case of three real singularities (i.e. the cubic consists of three real lines).

Example 4.2. Let us consider the cubic

$$f(x, y) = y^3 - 4yx^2 - x^3 + y^2 + 2xy - x^2 + \frac{17}{229}y + \frac{159}{229}x - \frac{14}{229} = 0$$

together with their partial derivatives with respect to y

$$f_y(x, y) = 3y^2 - 4x^2 + 2y + 2x + \frac{17}{229}$$

and

$$f_{yy}(x, y) = 6y + 2.$$

We now compute $R_f(x)$ and its squarefree decomposition:

$$R_f(x) = (52441x^3 - 41907x^2 + 10763x - 889)^2 = \left(R_{f,2}(x)\right)^2$$

with only one factor having three real roots x_1, x_2 and x_3 (so $m_1 = m_2 = m_3 = 2$). We choose r_1, r_2, r_3 and r_4 so that

$$r_1 < x_1 < r_2 < x_2 < r_3 < x_3 < r_4$$

and get $k_1 = k_2 = k_3 = k_4 = 3$. So we know our cubic is just three lines and the three singularities are crunodes.

Our method starts by substituting $y = -\frac{1}{3}$ (obtained after solving $f_{yy}(x, y) = 0$) in $f(x, y)$ and obtaining

$$g(x) = f\left(x, -\frac{1}{3}\right) = -x^3 + \frac{1}{3}x^2 + \frac{19}{687}x - \frac{73}{6183}.$$

It is easy to check that the first real root of $g(x)$ is negative while all the roots of R_f are positive (use for example Descartes method for $R_f(-x)$ and $g(-x)$ in order to get such a conclusion). So $V(f_{yy})$ cuts L to the left of the singularities. Now we go to infinity and compare the slope of $V(f_{yy})$, which is 0, with the real roots of

$$f_3(1, y) = y^3 - 4y - 1.$$

We substitute $y = 0$ in f_3 and get -1, which is negative. Therefore there are two real roots of $f_3(1, y)$ below 0, which means that L is below $V(f_{yy})$ after the intersection, i.e. during the events. Therefore, $y_i - y_i'$ is positive for $i = 2$ and negative for $i = 1$ and $i = 3$.

Fig. 2 shows the cubic $V(f)$ and $V(f_{yy})$ together with the topology of $V(f)$ as produced by the `Maple` implementation of the algorithm presented in the previous section.

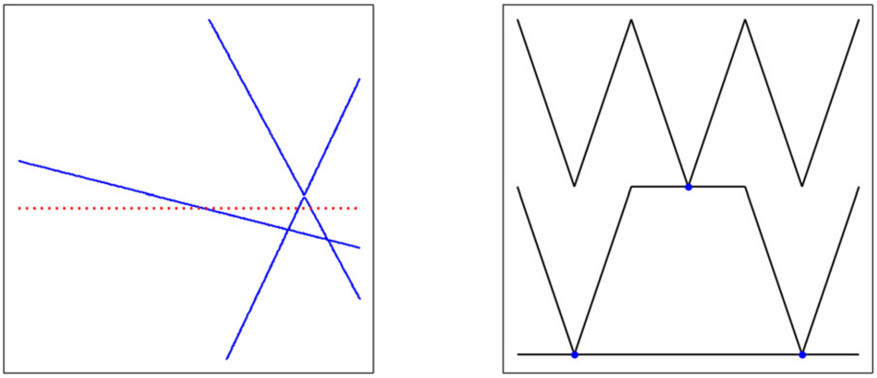

Fig. 2. The three lines example: $V(f)$ and $V(f_{yy})$ (left) and the topology of $V(f)$ (right).

Remark 4.1. In the previous example, none of the three lines in $V(f)$ has an equation with rational coefficients. So factoring f is not the best option to characterize the topology of $V(f)$ since such a factorization will involve the using of complex algebraic algorithms and algebraic numbers. For the concrete case considered in the previous example the three lines in $V(f)$ can be presented as

$$y+\left(-\frac{2\alpha^2}{109233}-\frac{11\alpha}{477}+\frac{43}{477}\right)x-\frac{2\alpha^2}{109233}+\frac{661\alpha}{109233}+\frac{202}{477}$$

where α denotes any real root of the polynomial

$$\Gamma(\alpha)=4\alpha^3-29541\alpha-209764.$$

The polynomial $\Gamma(\alpha)$ has three different real roots and this means that

$$f(x,y)=\prod_{\{\alpha\in\mathbb{R}:\ \Gamma(\alpha)=0\}}\left(y+\left(-\frac{2\alpha^2}{109233}-\frac{11\alpha}{477}+\frac{43}{477}\right)x-\frac{2\alpha^2}{109233}+\frac{661\alpha}{109233}+\frac{202}{477}\right).$$

In fact, asking to a Computer Algebra System like `Maple` for the factorization of $f(x,y)$ (and allowing algebraic numbers in the output) produces a much more complicated answer than the one previously presented.

When $f(x,y)\in\mathbb{Q}[x,y]$ is the union of a smooth conic and a line, it can be proved (see [4]) that both of them have rational coefficients but factoring f is more expensive than just checking what we have proposed in the previous section on two x-extremal points. In the parabola case, the study of the x-extremal point at the infinity line is also efficient enough and the case without x-extremal points involves very easy computations.

It must be noted that, in advance, it is not known the kind of cubic curve we are dealing with. Thus, it is clearly not a good option to begin by asking for the factorization of the considered polynomials that can also involve complicated algebraic numbers. With the methods we proposed, only operations with rational numbers are required.

Next example deals with the case of two singularities and two x-extremal points.

Example 4.3. Consider

$$f(x,y)=y^3+x^2y-y-x^3-xy^2+x.$$

Then

$$R_f(x)=16x^6-32x^4+20x^2-4=16(x^2-1)\left(x^2-\frac{1}{2}\right)^2=16R_{f,1}(x)\big(R_{f,2}(x)\big)^2.$$

Moreover, we get

$$r_1=-2\quad r_2=-\frac{3}{4}\quad r_3=0\quad r_4=\frac{3}{4}\quad r_5=2\quad k_1=1\quad k_2=3\quad k_3=3\quad k_4=3\quad k_5=1$$

together with $m_1=m_4=1$ and $m_2=m_3=2$.

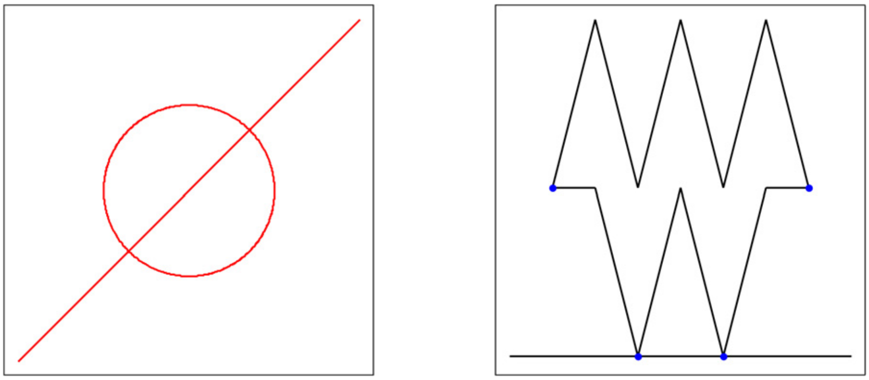

Fig. 3. The ellipse and line example: the topology of $V(f)$ (right).

This means that we have an ellipse (since the interval of the x-coordinates of the extremal points contains the interval of the x-coordinates of the singular points). When we analyze the x-extremal points, we get that the uninvolved arc is below the event in the left one and vice-versa on the right one.

To analyze the left singularity (which we know has to be a node because this is the only possibility to have two real singularities in a reducible cubic), we observe, from the information of the left x-extremal point, that the two upper arcs between the extremal point and the singularity belong to the ellipse. These arcs cannot intersect each other because an ellipse is smooth. Therefore, the intersection is between the two arcs below and hence the uninvolved arc is the upper one.

Fig. 3 shows the cubic $V(f)$ together with its topology as produced by the `Maple` implementation of the algorithm presented in the previous section.

Last example deals with the case of two singularities and no x-extremal points.

Example 4.4. Consider the cubic

$$f(x, y) = y^3 - y - x^2 y + x.$$

Then

$$R_f(x) = 4x^6 + 12x^4 - 15x^2 + 4 = 4(x^2 - 4)\left(x^2 - \frac{1}{2}\right)^2 = 4R_{f,1}(x)\big(R_{f,2}(x)\big)^2$$

and $r_1 = -4$, $r_2 = 0$ and $r_3 = 4$ with $m_1 = m_2 = 2$ and $k_1 = k_2 = k_3 = 3$.

We compute $f_{yy}(x, y) = 6y$ and intersect $V(f_{yy})$ with $V(f)$, which gives the only point $(0, 0)$ with multiplicity 1. All the events involving $V(f)$ or $V(f_{yy})$ are clearly between the singularities, so we choose r to be -4 for the left singularity and 4 for the right one. We compute $f(-4, 0) = -4$ and $f(4, 0) = 4$ and get that the intersections of the hyperbola with the line (i.e. the event points) are with the lower branch to the left and with the upper branch to the right.

Fig. 4 shows the cubic $V(f)$ together with its topology as produced by the `Maple` implementation of the algorithm presented in the previous section.

We end this section by showing the efficiency of the method we propose. Both algorithms, the one introduced in [6] and the one presented here, have been both implemented by the authors of this paper in the Computer Algebra System `Maple` and tested on a PowerPC G5 at 1.8 GHz with 1.25 GB of RAM. Both algorithms have been implemented in `Maple` in order to make a more proper comparison.

The new algorithm computes the topology of 100 randomly generated reducible cubics (since for the non-reducible cubics we do not propose any alternative to the algorithm presented in [6]) with integer coefficients between -10^2 and 10^2 in 23.06 seconds while the method in [6] required 28.48 seconds. When taking the coefficients between -10^8 and 10^8,

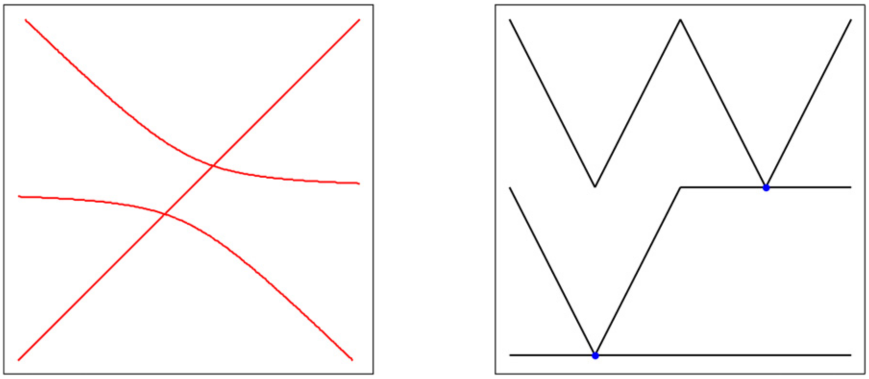

Fig. 4. The no extremal points example: the topology of $V(f)$ (right).

- the new algorithm computed the topology of 100 pairs of conic and line in 24.54 seconds while the algorithm in [6] required 31.83 seconds;
- the new algorithm computed the topology of 100 cubics consisting of three lines in 54.11 seconds while the algorithm in [6] required 81.15 seconds.

The method introduced in this paper improves the computation of the topology of a reducible cubic and this improvement comes from using deeper the well-known geometry of the reducible cubics instead of relying on general algebraic tools.

Acknowledgements

We extremely thank the two referees of this paper for their detailed and very useful comments and suggestions.

References

[1] E. Berberich, A. Eigenwillig, M. Hemmer, S. Hert, K. Mehlhorn, E. Schömer, A computational basis for conic arcs and boolean operations on conic polygons, in: Proceedings of the 10th European Symposium on Algorithms (ESA 2002), Lecture Notes in Computer Science, vol. 2461, 2002, pp. 174–186.

[2] S. Basu, R. Pollack, M.-F. Roy, Algorithms in Real Algebraic Geometry, Algorithms and Computation in Mathematics, vol. 10, Springer-Verlag, 2006.

[3] J. Caravantes, L. Gonzalez-Vega, Computing the topology of an arrangement of quartics, in: Mathematics of Surfaces XII, Lecture Notes in Computer Science, vol. 4647, 2007, pp. 104–120.

[4] J. Caravantes, L. Gonzalez-Vega, Efficiently factoring over the complex field algebraic plane curves of degree three with rational coefficients, Preprint, 2007.

[5] A. Eigenwillig, M. Kerber, N. Wolpert, Fast and exact geometric analysis of real algebraic plane curves, in: Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC'07), ACM Press, pp. 151–158.

[6] A. Eigenwillig, L. Kettner, E. Schömer, N. Wolpert, Exact, efficient and complete arrangement computation for cubic curves, Computational Geometry 35 (2006) 36–73.

[7] E. Flato, D. Halperin, I. Hanniel, O. Nechushtan, E. Ezra, The design and implementation of planar maps in CGAL, ACM Journal of Experimental Algorithmics 5 (2000), article 13.

[8] J. von zur Gathen, J. Gerhard, Modern Computer Algebra, Cambridge University Press, 1999.

[9] C.G. Gibson, Elementary Geometry of Algebraic Curves: An Undergraduate Introduction, Cambridge University Press, 1998.

[10] L. Gonzalez-Vega, I. Necula, Efficient topology determination of implicitly defined algebraic plane curves, Computer Aided Geometric Design 19 (2002) 719–743.

[11] Y.B. Li, A new approach for constructing subresultants, Applied Mathematics and Computation 183 (2006) 471–476.

[12] K. Mehlhorn, S. Noher, LEDA: A Platform for Combinatorial and Geometric Computing, Cambridge University Press, 1999.

[13] M. Seel, Implementation of planar Nef polyhedra, Report MPI-I-2001-1-003, Max-Planck-Institut für Informatik, 2001.

[14] R. Wein, High-level filtering for arrangements of conic arcs, in: Proceedings of the 10th European Symposium on Algorithms (ESA 2002), Lecture Notes in Computer Science, vol. 2461, 2002, pp. 884–895.